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One-Parameter Inhomogeneous Differential Realization and Boson-Fermion Realization of the SPL(2, 1) Superalgebra

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Abstract One-parameter inhomogeneous differential realization of the SPL(2, 1) superalgebra on the space of homogeneous polynomials and the corresponding boson-fermion realization are studied. The parameter α may be related to the interaction parameter U in one exactly solvable model for correlated electrons.

Keywords SPL(2, 1) superalgebra · Inhomogeneous differential realization · Boson-fermion realization · Exactly solvable model

1 Introduction

A series of models of correlated electrons on a lattice and exactly solvable in one dimension and supersymmetric, such as Hubbard and extended Hubbard models and t–J model, EKS model, BGLZ model [1], has been extensively studied due to their promising role in theoretical condensed-matter physics and possibly in high-T_c superconductivity. Those models contain one symmetry-preserving free real parameter which is the Hubbard interaction parameter U. Quasi-exactly solvable problems (QESP) in quantum mechanics have been discussed by Turbiner and Ushveridze [9]. QESP in quantum mechanics have become increasingly important because they have been generalized to the study of the conformal field theory. A connection of QESP and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) has been described at the first time by Turbiner [7, 8, 10]. The key resolving the QESP lies in studying finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. Some inhomogeneous differential realizations of Lie superalgebras SPL(2, 1) and GL(2|1) have been given by Chen [2–5]. One-parameter homogeneous differential realization and boson-fermion realization of the SPL(2, 1) superalgebra have been studied [6]. In the present paper we shall be concerned with the SPL(2, 1) superalgebra. The purpose of the present paper is to obtain One-parameter inhomogeneous

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differential realization of the SPL(2, 1) superalgebra and the corresponding boson-fermion realization. This paper is arranged as follows. In Sect. 2 by employing variable substitution we derive inhomogeneous differential realization of the SPL(2, 1) on the spaces of inhomogeneous polynomials. In Sect. 3 we consider their corresponding relations of C-number differential operators and boson creation and annihilation operators, of Grassmann number differential operators and fermion creation and annihilation operators respectively. The corresponding one-parameter inhomogeneous boson-fermion realization of the SPL(2, 1) superalgebra is obtained in terms of only one pair of boson operators and two pairs of fermion operators.

2 One-Parameter Inhomogeneous Differential Realization of the SPL(2, 1)

In accordance with Chen [2] the generators of the SPL(2, 1) superalgebra read as follows:

$$\{Q_3, Q_+, Q_-, B \in \mathrm{SPL}(2,1)_{\bar{0}} \mid V_+, V_-, W_+, W_- \in \mathrm{SPL}(2,1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{gathered}
[Q_3, Q_\pm] = \pm Q_\pm, \qquad [Q_+, Q_-] = 2Q_3, \qquad [B, Q_\pm] = [B, Q_3] = 0, \\
[Q_3, V_\pm] = \pm\frac{1}{2}V_\pm, \qquad [Q_3, W_\pm] = \pm\frac{1}{2}W_\pm, \qquad [B, V_\pm,] = \frac{1}{2}V_\pm, \\
[B, W_\pm] = -\frac{1}{2}W_\pm, \qquad [Q_\pm, V_\mp] = V_\pm, \qquad [Q_\pm, W_\mp] = W_\pm, \qquad [Q_\pm, V_\pm] = 0, \\
[Q_\pm, W_\pm] = 0, \qquad \{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = \{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0, \\
\{V_\pm, W_\pm\} = \pm Q_\pm, \qquad \{V_\pm, W_\mp\} = -Q_3 \pm B.
\end{gathered} \tag{2}$$

In Ref. [6] we have obtained one-parameter homogeneous differential realization of the SPL(2, 1) superalgebra Its explicit form as follows:

$$\begin{gathered}
Q_3 = \frac{1}{2}\left(\mu_1\frac{\partial}{\partial\mu_1} - \mu_2\frac{\partial}{\partial\mu_2}\right), \qquad Q_+ = \mu_1\frac{\partial}{\partial\mu_2}, \qquad Q_- = \mu_2\frac{\partial}{\partial\mu_1}, \\
B = \left(\frac{1}{2}+\alpha\right)\left(\mu_1\frac{\partial}{\partial\mu_1} + \mu_2\frac{\partial}{\partial\mu_2}\right) + \alpha\left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2}\right), \\
V_+ = \sqrt{\alpha}\mu_1\frac{\partial}{\partial\xi_1} + \sqrt{1+\alpha}\xi_2\frac{\partial}{\partial\mu_2}, \\
V_- = \sqrt{\alpha}\mu_2\frac{\partial}{\partial\xi_1} - \sqrt{1+\alpha}\xi_2\frac{\partial}{\partial\mu_1}, \\
W_+ = -\sqrt{\alpha}\xi_1\frac{\partial}{\partial\mu_2} + \sqrt{1+\alpha}\mu_1\frac{\partial}{\partial\xi_2}, \\
W_- = \sqrt{\alpha}\xi_1\frac{\partial}{\partial\mu_1} + \sqrt{1+\alpha}\mu_2\frac{\partial}{\partial\xi_2}.
\end{gathered} \tag{3}$$

In order to get one-parameter inhomogeneous differential realization of the SPL(2, 1) superalgebra on the space of inhomogeneous polynomials, we introduce three new independent variables (x, y_1, y_2) and employ variable substitution

$$x = \frac{\mu_1}{\mu_2}, \qquad y_1 = \frac{\xi_1}{\mu_2}, \qquad y_2 = \frac{\xi_2}{\mu_2}, \tag{4}$$

where x is a C-number and y_1, y_2 are Grassmann numbers respectively. Clearly, the basis of A_n becomes

$$\mu_1^{i_1}\mu_2^{i_2}\xi_1^{k_1}\xi_2^{k_2} \Rightarrow x^{i_1}\mu_2^n y_1^{k_1} y_2^{k_2} \quad (i_1 + k_1 + k_2 = 0, 1, \dots, n). \tag{5}$$

Let

$$\tilde{A}_{\mathrm{n}} = \{x^{i_1}\mu_2^n y_1^{k_1} y_2^{k_2} \mid i_1 + k_1 + k_2 = 0, 1, \dots, n,\ i_1 \in Z^+,\ k_1, k_2 = 0, 1\} \tag{6}$$

then $\tilde{A}_{\mathrm{n}}$ is a space of inhomogeneous polynomials.

Using (3), (4) and the following definition

$$\begin{aligned}\bar{F}(x^{i_1}\mu_2^n y_1^{k_1} y_2^{k_2}) &= (\hat{F}x^{i_1})\mu_2^n y_1^{k_1} y_2^{k_2} + x^{i_1}(\hat{F}\mu_2^n) y_1^{k_1} y_2^{k_2} \\ &\quad + x^{i_1}\mu_2^n(\hat{F}y_1^{k_1}) y_2^{k_2} + x^{i_1}\mu_2^n y_1^{k_1}(\hat{F}y_2^{k_2})\end{aligned} \tag{7}$$

we get one-parameter inhomogeneous differential realization $\bar{F}$ of the SPL(2, 1) on $\tilde{A}_n$,

$$\begin{aligned}
\bar{Q}_3 &= -\frac{1}{2}n + x\frac{\partial}{\partial x} + \frac{1}{2}y_1\frac{\partial}{\partial y_1} + \frac{1}{2}y_2\frac{\partial}{\partial y_2}, \\
\bar{Q}_+ &= nx - x^2\frac{\partial}{\partial x} - xy_1\frac{\partial}{\partial y_1} - xy_2\frac{\partial}{\partial y_2}, \\
\bar{Q}_- &= \frac{\partial}{\partial x}, \\
\bar{B} &= \left(\frac{1}{2} + \alpha\right)n - \frac{1}{2}y_1\frac{\partial}{\partial y_1} - \frac{1}{2}y_2\frac{\partial}{\partial y_2}, \\
\bar{V}_+ &= \sqrt{1+\alpha}\,ny_2 + \sqrt{\alpha}\,x\frac{\partial}{\partial y_1} - \sqrt{1+\alpha}\,y_2x\frac{\partial}{\partial x} - \sqrt{1+\alpha}\,y_2y_1\frac{\partial}{\partial y_1}, \\
\bar{V}_- &= -\sqrt{1+\alpha}\,y_2\frac{\partial}{\partial x} + \sqrt{\alpha}\frac{\partial}{\partial y_1}, \\
\bar{W}_+ &= -\sqrt{\alpha}\,ny_1 + \sqrt{1+\alpha}\,x\frac{\partial}{\partial y_2} + \sqrt{\alpha}\,y_1x\frac{\partial}{\partial x} + \sqrt{\alpha}\,y_1y_2\frac{\partial}{\partial y_2}, \\
\bar{W}_- &= \sqrt{\alpha}\,y_1\frac{\partial}{\partial x} + \sqrt{1+\alpha}\frac{\partial}{\partial y_2}.
\end{aligned} \tag{8}$$

It is worthy of note that μ_2 is a cofactor in the basis of $\tilde{A}_n$. Granted that we extend the non-negative integer n to any real number, one still gets (8). It is easily proved that the generators thus represented satisfy all the commutation and anticommutation relations of the SPL(2, 1).

3 One-Parameter Inhomogeneous Boson-Fermion Realization of the SPL(2, 1)

Considering their corresponding relations of C-number differential operators $(x, \frac{\partial}{\partial x})$ and boson creation and annihilation operators (b^+, b),

$$b^+ \Leftrightarrow x, \qquad b \Leftrightarrow \frac{\partial}{\partial x}, \qquad [b, b^+] = 1, \qquad \left[\frac{\partial}{\partial x}, x\right] = 1, \tag{9}$$

and of Grassmann number differential operators $(y_i, \frac{\partial}{\partial y_i})$ and fermion creation and annihilation operators (a_i^+, a_i), respectively,

$$\begin{aligned} &a_i^+ \Leftrightarrow y_i, \qquad a_i \Leftrightarrow \frac{\partial}{\partial y_i}, \qquad \{a_i, a_j^+\} = \delta_{ij}, \qquad \left\{\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right\} = \delta_{ij}, \\ &\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0, \qquad \left\{\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right\} = \{y_i, y_j\} = 0 \end{aligned} \tag{10}$$

the corresponding inhomogeneous boson-fermion realization of the SPL(2, 1) is obtained in terms of one pair of boson operators and two pairs of fermion operators as follows:

$$\begin{aligned} \tilde{Q}_3 &= -\frac{1}{2}n + b^+b + \frac{1}{2}a_1^+a_1 + \frac{1}{2}a_2^+a_2, \\ \tilde{Q}_+ &= nb^+ - b^{+2}b - b^+a_1^+a_1 - b^+a_2^+a_2, \\ \tilde{Q}_- &= b, \\ \tilde{B} &= \left(\frac{1}{2} + \alpha\right)n - \frac{1}{2}a_1^+a_1 - \frac{1}{2}a_2^+a_2, \\ \tilde{V}_+ &= \sqrt{1+\alpha}\,na_2^+ + \sqrt{\alpha}\,b^+a_1 - \sqrt{1+\alpha}\,a_2^+b^+b - \sqrt{1+\alpha}\,a_2^+a_1^+a_1, \\ \tilde{V}_- &= -\sqrt{1+\alpha}\,a_2^+b + \sqrt{\alpha}\,a_1, \\ \tilde{W}_+ &= -\sqrt{\alpha}\,na_1^+ + \sqrt{1+\alpha}\,b^+a_2 + \sqrt{\alpha}\,a_1^+b^+b + \sqrt{\alpha}\,a_1^+a_2^+a_2, \\ \tilde{W}_- &= \sqrt{\alpha}\,a_1^+b + \sqrt{1+\alpha}\,a_2. \end{aligned} \tag{11}$$

Obviously, we use only one pair of boson operators and two pairs of fermion operators in obtaining one-parameter inhomogeneous boson-fermion realization.

4 Conclusion

We have obtained one-parameter inhomogeneous differential realization and the corresponding boson-fermion realization of the SPL(2, 1) superalgebra In terms of the conclusion it may be of use for further researches on one-parameter indecomposable and irreducible representations of the SPL(2, 1) superalgebra and for determining new concrete structure of the quasi-exactly-solvable Hamiltonian corresponding to the SPL(2, 1) supersymmetrical quantum system.

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